

Polynomial Birack Modules

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Abstract

Birack modules are modules over an algebra $\mathbb{Z}[X]$ associated to a finite birack X . In previous work, birack module structures on \mathbb{Z}_n were used to enhance the birack counting invariant. In this paper, we use birack modules over Laurent polynomial rings $\mathbb{Z}_n[q^{\pm 1}]$ to enhance the birack counting invariant, defining a customized Alexander polynomial-style signature for each X -labeled diagram; the multiset of these polynomials is an enhancement of the birack counting invariant. We provide examples to demonstrate that the new invariant is stronger than the unenhanced birack counting invariant and is not determined by the generalized Alexander polynomial.

KEYWORDS: Biracks, Birack Modules, Alexander Polynomial, Sawollek Polynomial, Generalized Alexander Polynomial, Enhancements of Counting Invariants

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1 Introduction

Biracks are algebraic structures with axioms motivated by framed oriented Reidemeister moves. They were introduced in [6] in order to define invariants of framed oriented knots and links. In [11] a property of finite biracks called *birack rank* was used to define a computable integer-valued invariant of unframed oriented knots and links called the *integral birack counting invariant*, $\Phi_X^{\mathbb{Z}}$. In [1] an algebra $\mathbb{Z}[X]$ called the *rack algebra* was associated to a finite rack X (a particular type of birack), and modules over $\mathbb{Z}[X]$ were studied. In [7] rack module structures over \mathbb{Z}_n were used to enhance the rack counting invariant, defining a new invariant Φ_X^M which specializes to the integral rack counting invariant $\Phi_X^{\mathbb{Z}}$ but is generally stronger. In [2] rack modules were generalized to the case of biracks and birack modules over \mathbb{Z}_n were employed to define enhancements of the birack counting invariant.

In this paper we consider birack modules over Laurent polynomial rings $\mathbb{Z}_n[q^{\pm 1}]$; such a birack module lets us define a customized Alexander polynomial for each birack homomorphism $f : BR(L) \rightarrow X$, the multiset of which forms a new enhancement of the integral counting invariant. Moreover, the Generalized Alexander (Sawollek) and classical Alexander polynomials emerge as special cases of the enhanced invariant.

The paper is organized as follows. In Section 2 we review the basics of biracks and birack modules. In Section 3 we define the enhanced link invariant. In Section 4 we collect some examples and in Section 5 we end with some questions for future work.

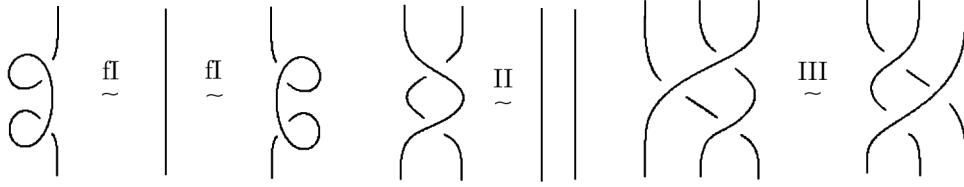
2 Biracks and birack modules

We begin with a definition. First introduced in [6], a *birack* is an algebraic structure consisting of a set X and a map $B : X \times X \rightarrow X \times X$ with axioms derived from the *oriented framed Reidemeister moves*, obtained

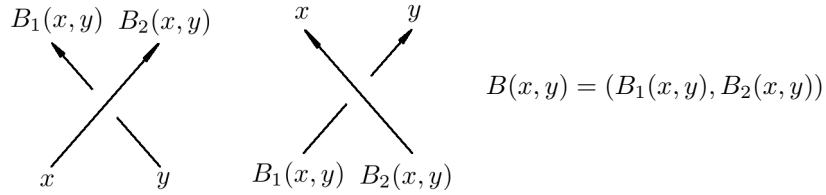
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by considering all ways of orienting the strands in the moves



with the correspondence (also known as the *semiarc labeling rule*)



In [9] and later in [5] the unframed oriented case, known as the *strong biquandle* case, was considered; the version below comes from [11].

Definition 1 Let X be a set and $\Delta : X \rightarrow X \times X$ the diagonal map defined by $\Delta(x) = (x, x)$. Then an invertible map $B : X \times X \rightarrow X \times X$ is a *birack map* if the following conditions are satisfied:

- (i) There exists a unique invertible map $S : X \times X \rightarrow X \times X$ called the *sideways map* such that for all $x, y \in X$, we have

$$S(B_1(x, y), x) = (B_2(x, y), y).$$

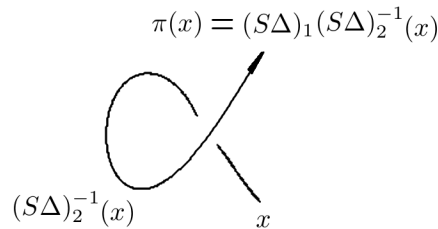
- (ii) The components $(S^{\pm 1}\Delta)_{1,2} : X \rightarrow X$ of the composition of the sideways map and its inverse with the diagonal map Δ are bijections, and

- (iii) B satisfies the *set-theoretic Yang-Baxter equation*

$$(B \times I)(I \times B)(B \times I) = (I \times B)(B \times I)(I \times B).$$

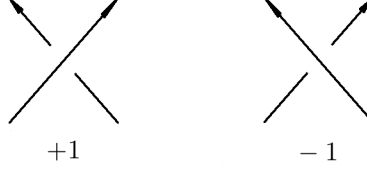
These axioms are the conditions required to ensure that each labeling of an oriented framed knot or link diagram according to the semiarc labeling rule above before a framed oriented Reidemeister move corresponds to a unique such labeling after the move. By construction, the number of labelings of an oriented framed link diagram is an invariant of framed isotopy, called the *basic birack counting invariant*, denoted Φ_X^B .

Let X be a birack. The bijection $\pi : X \rightarrow X$ defined by $\pi = (S\Delta)_1(S\Delta)_2^{-1}$ represents going through a positive kink:



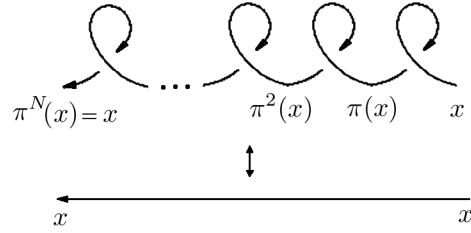
This bijection π , called the *kink map*, is an element of the symmetric group on X ; if X is finite, then π has a finite exponent $N \in \mathbb{Z}$ such that $\pi^N(x) = x$ for all $x \in X$. We call N the *birack rank* or *birack characteristic* of X . The map $(S\Delta)_2^{-1}$ is sometimes called α . A birack with rank $N = 1$ is a *strong biquandle*.

Recall that the framing number of a component of a blackboard framed link is given by the *writhe* of the component, i.e., the number of positive self-crossings of the component minus the number of negative self-crossings.



On a link of c components, we can specify framings with framing vectors $\vec{w} \in \mathbb{Z}^c$ where the k th component of \vec{w} is the number of times the k th component crosses itself at a positive crossing minus the number of times the k th component crosses itself at a negative crossing.

If X is a finite birack with rank N , then labelings of a link by X before and after an N -phone cord move



are in bijective correspondence. Thus, for any oriented link L of c components, the c -dimensional lattice of basic counting invariant values $\Phi_X^B(L, \vec{w})$ is tiled with a tile consisting of framing vectors in $(\mathbb{Z}_N)^c$. The sum of basic counting invariants over one such tile is an invariant of the unframed oriented link known as the *integral birack counting invariant*,

$$\Phi_X^{\mathbb{Z}} = \sum_{\vec{w} \in (\mathbb{Z}_N)^c} \Phi_X^B(L, \vec{w}).$$

In particular, we have

Theorem 1 *If X is a finite birack and L, L' are ambient isotopic oriented links, then $\Phi_X^{\mathbb{Z}}(L) = \Phi_X^{\mathbb{Z}}(L')$.*

See [11] for more.

We can specify a birack map on a finite set $X = \{x_1, x_2, \dots, x_n\}$ with a *birack matrix* specifying the operation tables of the two component maps $B_1, B_2 : X \times X \rightarrow X$ of B considered as binary operations. More precisely, the birack matrix of X is an $n \times 2n$ block matrix $[U|L]$ with $U_{j,k} = l$ and $L_{j,k} = m$ where $x_l = B_1(x_k, x_j)$ and $x_m = B_2(x_j, x_k)$. Note the reversed order of the inputs in B_1 ; the notation is chosen so that the output label and the input row label are part of the same strand. It is sometimes convenient to abbreviate $B_1(x, y) = y^x$ and $B_2(x, y) = x_y$, i.e. $B(x, y) = (y^x, x_y)$.

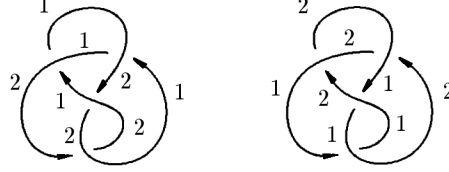
Example 1 Let $X = \{x_1, x_2\}$ be the birack with birack matrix

$$M = \left[\begin{array}{cc|cc} 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \end{array} \right].$$

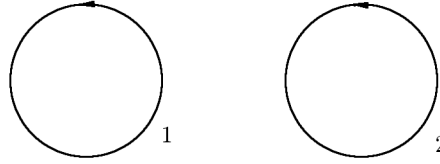
We can interpret X in this example as a labeling rule saying that when a strand crosses under another strand, it keeps the same label, but a strand crossing over another strand switches labels from 1 to 2 or from 2 to 1. The kink map π is then the transposition (12), so this birack has rank $N = 2$.

To compute the integral birack counting invariant $\Phi_X^{\mathbb{Z}}$ for a link L and birack X of rank N , we need to consider diagrams of L with all framing vectors in $(\mathbb{Z}_N)^c$. For a knot K and our birack X , this means

finding all labelings of one diagram of K with an even writhe and one diagram of K with an odd writhe. For example, the even-writhe figure eight knot has two X -labelings, while the odd-writhe figure eight knot has no valid X -labelings.



Similarly, the even-writhe unknot has two X -labelings and the odd-writhe unknot has no X -labelings.



For a link of two components, we would need to count labelings for a diagram of L with both writhes even, one with both writhe odd, and the two even-odd writhe combinations, etc.

The integral birack counting invariant Φ_X^Z in example 1 does not distinguish the figure eight knot from the unknot, but these X -labeled knot diagrams are still apparently quite different. Thus, we would like to develop a way to tell the birack-labeled links apart. An invariant of birack-labeled links is called a *signature* of the labeled link; collecting the signatures of the labelings of a link over a complete $(\mathbb{Z}_N)^c$ tile defines an *enhancement* of the integral birack counting invariant. In [2] an enhancement of the birack counting invariant was defined using *birack modules*:

Definition 2 Let X be a finite birack and let Λ be the polynomial ring $\Lambda = \mathbb{Z}[t_{x,y}^{\pm 1}, s_{x,y}, r_{x,y}^{\pm 1}]$ with invertible variables $t_{x,y}, r_{x,y}$ and generic variables $s_{x,y}$ indexed by ordered pairs of birack elements $x, y \in X$. The *birack algebra* associated to X is the quotient algebra $\mathbb{Z}[X] = \Lambda/I$ where I is the ideal generated by elements of the forms

$$\begin{aligned} & \bullet \quad r_{x,y,z} r_{x,y} - r_{x_{zy}, y_z} r_{x,z^y} & \bullet \quad t_{x_{zy}, y_z} r_{y,z} - r_{y^x, z^{xy}} t_{x,y} & \bullet \quad s_{x_{zy}, y_z} r_{x,z^y} - r_{y^x, z^{xy}} s_{x,y} \\ & \bullet \quad t_{x,z^y} t_{y,z} - t_{y^x, z^{xy}} t_{x_{yz}, z} & \bullet \quad t_{x,z^y} s_{y,z} - s_{y^x, z^{xy}} t_{x,y} & \bullet \quad s_{x,z^y} - t_{y^x, z^{xy}} s_{x_{yz}, z} r_{x,y} - s_{y^x, z^{xy}} s_{x,y} \\ & \bullet \quad 1 - \prod_{k=0}^{N-1} (t_{\pi^k(x), \alpha(\pi^k(x))} r_{\pi^k(x), \alpha(\pi^k(x))} + s_{\pi^k(x), \alpha(\pi^k(x))}) \end{aligned}$$

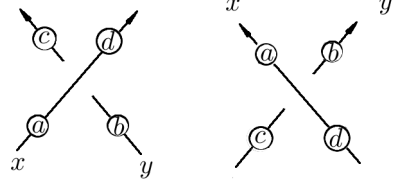
for $x, y, z \in X$ where we have $B(x, y) = (y^x, x_y)$.

If \mathbf{R} is a commutative ring, we can give \mathbf{R} the structure of a $\mathbb{Z}[X]$ -module by choosing elements $t_{x,y}, s_{x,y}, r_{x,y}$ of \mathbf{R} such that the ideal I in \mathbf{R} is zero. We can specify such a structure with an $n \times 3n$ block matrix $M_{\mathbf{R}} = [T|S|R]$ whose entries $T_{i,j} = t_{x_i, x_j}, S_{i,j} = s_{x_i, x_j}$ and $R_{i,j} = r_{x_i, x_j}$ make each of the generators of I zero.

Example 2 The birack X with matrix $M_X = \left[\begin{array}{cc|cc} 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \end{array} \right]$ has $\mathbb{Z}[X]$ -modules over the ring $\mathbf{R} = \mathbb{Z}_5$ including the module M given by the matrix

$$M_{\mathbf{R}} = \left[\begin{array}{cc|cc|cc} 1 & 1 & 2 & 1 & 2 & 2 \\ 1 & 1 & 4 & 2 & 3 & 3 \end{array} \right].$$

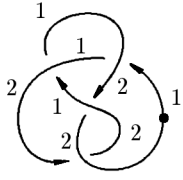
Given an X -labeling f of an oriented framed link diagram L and a $\mathbb{Z}[X]$ -module M , we can give the semiarcs in L a secondary labeling by elements of M , usually visualized as beads:



$$\begin{aligned} c &= t_{x,y}b + s_{x,y}a \\ d &= r_{x,y}a. \end{aligned}$$

The ideal I in definition 2 is chosen so that bead labelings before and after X -labeled framed Reidemeister and N -phone cord moves are in one-to-one correspondence. For each X -labeling, the set of bead labelings by M forms a $\mathbb{Z}[X]$ -module with presentation matrix determined by the crossing relations, called the *fundamental $\mathbb{Z}[X]$ -module* of f , denoted $\mathbb{Z}[f]$. Replacing the variables $t_{x,y}, s_{x,y}, r_{x,y}$ with their values in M yields the *fundamental M -module* of the birack labeled diagram L , denoted $M[f]$. By construction, X -labeled framed Reidemeister and N -phone cord moves induce isomorphisms of $M[f]$.

Example 3 The X -labeled figure eight knot below has fundamental $\mathbb{Z}[\mathbb{X}]$ -module and fundamental M -module given by the listed presentation matrices where the semiarcs are numbered starting with the pictured basepoint following the orientation. Replacing the variables $t_{x,y}, s_{x,y}$ and $r_{x,y}$ with their values in the X -module in example 2 yields the following matrix with entries in $M = \mathbb{Z}_5$:



$$\begin{array}{c|c} \mathbb{Z}[f] & M[f] \\ \hline \begin{bmatrix} -1 & t_{2,1} & 0 & 0 & 0 & 0 & s_{2,1} & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & r_{2,1} & 0 \\ 0 & 0 & s_{2,1} & 0 & -1 & t_{2,1} & 0 & 0 \\ 0 & -1 & r_{2,1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{2,2} & 0 & 0 & t_{2,2} & -1 \\ 0 & 0 & 0 & r_{2,2} & -1 & 0 & 0 & 0 \\ 0 & 0 & t_{2,2} & -1 & 0 & 0 & 0 & s_{2,2} \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & r_{2,2} \end{bmatrix} & \begin{bmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 3 & 0 \\ 0 & 0 & 4 & 0 & 4 & 1 & 0 & 0 \\ 0 & 4 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 0 & 0 & 0 & 2 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \end{array}$$

Row-reduction over \mathbb{Z}_5 yields a 3-dimensional solution space, so there are $5^3 = 25$ total M -labelings of the pictured X -labeled diagram.

In [2], the *birack module enhanced invariant* associated to a pair (X, M) of a birack X and birack module M was defined as

$$\Phi_X^M(L) = \sum_{f \in \mathcal{L}(L, X)} u^{|M[f]|}$$

where $\mathcal{L}(L, X)$ is a set of X -labelings over a complete tile of framings of L modulo the rank rank N of X .

Example 4 The birack module enhanced invariant associated to the birack and module in example 3 distinguishes the figure eight knot 4_1 from the unknot 0_1 with $\Phi_X^M(4_1) = 2u^{25} \neq 2u^5 = \Phi_X^M(0_1)$.

3 Polynomial birack modules and an enhanced link invariant

Let \mathbf{R} be a Laurent polynomial ring and let M be a module over \mathbf{R} with presentation matrix A , i.e., $A \in M_{n,m}(\mathbf{R})$ such that $M = \text{Ker}(A)$. The k th elementary ideal I_k of M is the ideal $I_k \subset \mathbf{R}$ generated by the $(n - k)$ -minors of A . It is well-known (see for example [10]) that I_k does not depend on the choice of

presentation matrix A for M . In particular, the greatest common divisor of the $(n - k)$ -minors of A with respect to a fixed choice of term ordering, denoted

$$\Delta_k(M) = \gcd\{(n - k)\text{-minors of } A\}$$

is a generator for the minimal principal ideal P_k containing I_k . If A is a square matrix, then $\Delta_0(M)$ is simply the determinant of M .

Definition 3 Let X be a finite birack for rank N , \mathbf{R} a Laurent polynomial ring and M a $\mathbb{Z}[X]$ -module structure on \mathbf{R} . The k th *polynomial birack module enhanced invariant* of an oriented link L is the multiset

$$\Phi_X^{M, \Delta_k}(L) = \{\Delta_k(M[f]) \mid f \in \mathcal{L}(L, X)\}$$

where $\mathcal{L}(L, X)$ is the set of X -labelings of diagrams of L over a complete tile of framings mod N .

We note that Φ_X^{M, Δ_k} extends to virtual knots and links in the usual way, i.e., by ignoring the virtual crossings. See [8] for more. By construction, we have:

Theorem 2 *If L and L' are ambient isotopic classical or virtual links, X is a finite birack, and M is a polynomial birack module, then for each $k = 0, 1, 2, \dots$ we have*

$$\Phi_X^{M, \Delta_k}(L) = \Phi_X^{M, \Delta_k}(L').$$

Remark 1 Note that, like the usual Alexander polynomial, the polynomials in Φ_X^{M, Δ_k} are only defined up to multiplication by units in \mathbf{R} . We can normalize these polynomials by multiplying by units to obtain a polynomial with constant term 1 for ease of comparison. For instance, if $\mathbf{R} = \mathbb{Z}_5[q^{\pm 1}]$, then the polynomial $4q^2 + 3q^3 + q^4$ would normalize to $1 + 2q + 4q^2$ after multiplication by $4q^{-2}$.

Example 5 Let $X = \{1\}$ be the birack of one element and $\mathbf{R} = \mathbb{Z}[t^{\pm 1}, r^{\pm 1}]$. Then X has a birack module structure given by the matrix

$$M_{\mathbf{R}} = \begin{bmatrix} t & 1 - tr & r \end{bmatrix}.$$

The 0th polynomial birack module enhanced invariant Φ_X^{M, Δ_0} is then a singleton set whose entry is the *generalized Alexander polynomial*, also known as the *Sawollek polynomial* after a different normalization; see [9, 12] for more.

Example 6 Let $X = \{1\}$ be the birack of one element and $\mathbf{R} = \mathbb{Z}[t^{\pm 1}]$. The \mathbf{R} -module structure with matrix

$$M_{\mathbf{R}} = \begin{bmatrix} t & 1 - t & 1 \end{bmatrix}$$

has $k = 1$ polynomial birack module enhanced invariant Φ_X^{M, Δ_1} a singleton set whose entry is the usual Alexander polynomial $\Delta(K)$. Indeed, Φ_X^{M, Δ_k} in this case has single entry given by the usual k th Alexander polynomial $\Delta_k(K)$.

Thus, we can think of a polynomial birack module structure on a ring R as determining a customized Alexander polynomial for each X -labeling of our link L , and the invariant Φ_X^{M, Δ_k} collects these polynomials to form a new invariant whose cardinality is the integral birack counting invariant but whose entries can distinguish knots and link which have the same integral birack counting invariant.

4 Computations and Applications

In this section we collect a few examples of the new invariant. We begin with an illustration of how the invariant is computed.

Example 7 Let X be the birack with birack matrix

$$X = \left[\begin{array}{cc|cc} 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \end{array} \right]$$

and let $\mathbf{R} = \mathbb{Z}_5[q^{\pm 1}]$. A computer search using our custom `python` code found birack modules over \mathbf{R} including the module specified by the matrix

$$M_R = \left[\begin{array}{cc|cc|cc} q & q & 1+2q & 2+4q & 1 & 1 \\ q & q & 3+q & 1+2q & 4 & 4 \end{array} \right].$$

The birack rank of X is 2, so for any link L of c components, we must find X -labelings for all writhe vectors in $(\mathbb{Z}_2)^c$. The virtual trefoil, denoted 2.1 in the virtual knot table on the knot atlas [3], has two X -labelings f_1, f_2 for even writhe and no X -labelings for odd writhe. These labelings determine the listed presentation matrices for the modules of bead labelings:

$$\begin{array}{ccc} \begin{array}{c} \text{Diagram 1: Virtual trefoil with labels } x, y, z, w. \text{ Writhe vector } (1, 1, 1, 1). \end{array} & \Rightarrow & \begin{bmatrix} s_{11} & 0 & t_{11} & -1 \\ r_{11} & -1 & 0 & 0 \\ -1 & s_{21} & 0 & t_{21} \\ 0 & r_{21} & 0 & -1 \end{bmatrix} \Rightarrow M_{f_1} = \begin{bmatrix} 1+2q & 0 & q & 4 \\ 1 & 4 & 0 & 0 \\ 4 & 3+q & 0 & q \\ 0 & 4 & 0 & 4 \end{bmatrix} \\ \\ \begin{array}{c} \text{Diagram 2: Virtual trefoil with labels } x, y, z, w. \text{ Writhe vector } (2, 2, 2, 2). \end{array} & \Rightarrow & \begin{bmatrix} s_{22} & 0 & t_{22} & -1 \\ r_{22} & -1 & 0 & 0 \\ -1 & s_{12} & 0 & t_{12} \\ 0 & r_{12} & 0 & 4 \end{bmatrix} \Rightarrow M_{f_2} = \begin{bmatrix} 1+2q & 0 & q & 4 \\ 4 & 4 & 0 & 0 \\ 4 & 2+4q & 0 & q \\ 0 & 1 & 0 & 4 \end{bmatrix} \end{array}$$

so the $k = 0$ invariant after normalization is

$$\Phi_X^{M, \Delta_0} = \{\det(M_{f_1}), \det(M_{f_2})\} = \{1 + q + 3q^2, 1 + q + 3q^2\} = \{2 \times (1 + q + 3q^2)\}.$$

Note that since $\Phi_X^{M, \Delta_0}(0_1) = \{2 \times 0\}$, this example shows that Φ_X^{M, Δ_0} is not determined by the integral birack counting invariant and hence is a proper enhancement.

Our next example demonstrates that Φ_X^{M, Δ_0} is not determined by the generalized Alexander polynomial by distinguishing two virtual knots which have the same generalized Alexander polynomial.

Example 8 Let X, R and M be as in example 7. Then the virtual knots numbered 4.10 and 4.17 in the knot atlas [3] both have generalized Alexander polynomial $(1+r)(1-r)(1-t)(1-rt)$, but are distinguished by Φ_X^{M, Δ_0} :



$$\Phi_X^{M, \Delta_0}(4.10) = \{2 \times (1 + 2q + 4q^2 + 3q^3)\} \quad \Phi_X^{M, \Delta_0}(4.17) = \{2 \times (1 + q + 3q^2)\}$$

Our final example gives an impression of the effectiveness of Φ_x^{M,Δ_k} as an invariant by sampling the ability of Φ_x^{M,Δ_k} to differentiate knots for a randomly selected birack module.

Example 9 Let X be the birack in example 7. We randomly selected a polynomial birack X -module over $R = \mathbb{Z}[q^{\pm 1}]$, given by the matrix

$$M_R = \left[\begin{array}{cc|cc|cc} q & q & 1+q & 3+3q & 2 & 2 \\ q & q & 2+2q & 1+q & 3 & 3 \end{array} \right]$$

and computed Φ_X^{M,Δ_0} for the virtual knots with 4 and fewer crossings as listed on the knot atlas ([3]) using our python code, available at www.esotericka.org. The results are collected in the following table.

| $\Phi_X^{M,\Delta_0}(L)$ | L |
|---|--|
| $\{2 \times (0)\}$ | 3.1, 3.5, 3.6, 3.7, 4.8, 4.10, 4.16, 4.32, 4.41, 4.47, 4.50, 4.55, 4.56, 4.58, 4.59, 4.68, 4.70, 4.71, 4.72, 4.75, 4.76, 4.77, 4.85, 4.86, 4.89, 4.90, 4.96, 4.98, 4.99, 4.102, 4.105, 4.106, 4.107, 4.108 |
| $\{2 \times (1 + q + 2q^2 + 4q^3 + 2q^4)\}$ | 4.29 |
| $\{2 \times (1 + q + 4q^2 + 4q^3)\}$ | 4.6, 4.13, 4.17, 4.19, 4.23, 4.24, 4.26, 4.31, 4.35, 4.42, 4.46, 4.51, 4.57, 4.66, 4.67, 4.79, 4.93, 4.97, 4.103 |
| $\{2 \times (1 + q + 4q^3 + 4q^4)\}$ | 4.9, 4.15, 4.37, 4.45, 4.69, 4.78, 4.92, 4.95, 4.104 |
| $\{2 \times (1 + 2q + 4q^2 + 3q^3)\}$ | 4.74 |
| $\{2 \times (1 + 2q + 3q^3 + 4q^4)\}$ | 4.12, 4.21, 4.36, 4.61, 4.65, 4.73 |
| $\{2 \times (1 + 3q + 4q^2 + 2q^3)\}$ | 4.30 |
| $\{2 \times (1 + 4q + 4q^2 + q^3)\}$ | 4.3, 4.25 |
| $\{2 \times (1 + 4q^2)\}$ | 3.2, 3.3, 4.1, 4.4, 4.5, 4.11, 4.14, 4.18, 4.20, 4.22, 4.27, 4.28, 4.33, 4.34, 4.38, 4.39, 4.40, 4.43, 4.44, 4.48, 4.49, 4.52, 4.54, 4.60, 4.62, 4.63, 4.64, 4.81, 4.82, 4.84, 4.87, 4.88, 4.94, 4.101 |
| $\{2 \times (1 + 3q^2 + q^4)\}$ | 4.2 |
| $\{2 \times (1 + 4q^4)\}$ | 4.7, 4.53, 4.80, 4.91, 4.100 |

5 Questions

In this section we collect a few questions and directions for future work.

The Alexander and generalized Alexander polynomial satisfy the well-known Conway skein relation. For which biracks X , modules M and nonnegative integers k do the elements of Φ_X^{M,Δ_k} satisfy a skein relation?

The examples we have selected for inclusion in this paper use biracks and polynomial birack modules of small cardinality for speed of computation and convenience of presentation; even so, the Φ_X^{M,Δ_k} invariant appears to be quite effective at distinguishing virtual knots with only these small M and X . We anticipate that biracks of larger cardinality and polynomial birack modules with more variables or over larger rings should be even more effective at distinguishing knots and links. Thus, fast algorithms for computing Φ_X^{M,Δ_k} for larger k , M and X are of interest.

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